Electronic Journal of Probability

Vol. 7 (2002) Paper no. 19, pages 1–19.

Journal URL
http://www.math.washington.edu/~ejpecp/
Paper URL
http://www.math.washington.edu/~ejpecp/EjpVol7/paper19.abs.html

TRANSIENCE AND NON-EXPLOSION OF CERTAIN STOCHASTIC NEWTONIAN SYSTEMS

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Abstract: We give sufficient conditions for non-explosion and transience of the solution (x_t, p_t) (in dimensions $\geqslant 3$) to a stochastic Newtonian system of the form

$$\begin{cases} dx_t = p_t\,dt \\ dp_t = -\frac{\partial V(x_t)}{\partial x}\,dt - \frac{\partial c(x_t)}{\partial x}\,d\xi_t \end{cases},$$

where $\{\xi_t\}_{t\geqslant 0}$ is a d-dimensional Lévy process, $d\xi_t$ is an Itô differential and $c \in C^2(\mathbb{R}^d, \mathbb{R}^d)$, $V \in C^2(\mathbb{R}^d, \mathbb{R})$ such that $V \geqslant 0$.

Keywords and phrases: Stochastic Newtonian systems; Lévy processes; α-stable Lévy processes; Transience; Non-explosion

AMS subject classification (2000): 60H10, 60G51, 60J75, 70H99, 60J35

Submitted to EJP on June 14, 2002. Final version accepted on October 2, 2002.

1 Introduction

This work contributes to the series of papers [13, 15], [3, 4], [6], [20] and [19] which are devoted to the qualitative study of the Newton equations driven by random noise. For related results see also [5], [23], [26, 27], [1], [22] and the references given there. Newton equations of this type are interesting in their own right: as models for the dynamics of particles moving in random media (cf. [25]), in the theory of interacting particles (cf. [28], [29]) or in the theory of random matrices (cf. [24]), to mention but a few. On the other hand, the study of these equations fits nicely into the the larger context of (stochastic) partial differential equations, in particular Hamilton-Jacobi, heat and Schrödinger equations, driven by random noise (see [32, 33] and [14, 16, 17, 18]).

In most papers on this subject the driving stochastic process is a diffusion process with continuous sample paths, usually a standard Wiener process. Motivated by the recent growth of interest in Lévy processes, which can be observed both in mathematics literature and in applications, the present authors started in [20] and [19] the analysis of Newton systems driven by jump processes, in particular symmetric stable Lévy processes. In [20] we studied the rate of escape of a "free" particle driven by a stable Lévy process and its applications to the scattering theory of a system describing a particle driven by a stable noise and a (deterministic) external force.

In this paper we study non-explosion and transience of Newton systems of the form

$$\begin{cases} dx_t = p_t\,dt \\ dp_t = -\dfrac{\partial V(x_t)}{\partial x}\,dt - \dfrac{\partial c(x_t)}{\partial x}\,d\xi_t \end{cases}, \tag{1}$$

where $\xi_t = (\xi_t^1, \dots, \xi_t^d)$ is a d-dimensional Lévy process, $c \in C^2(\mathbb{R}^d, \mathbb{R}^d)$, $V \in C^2(\mathbb{R}^d)$, $V \geqslant 0$ and $\left(\frac{\partial c(x_t)}{\partial x}\,d\xi_t\right)_i := \sum_{j=1}^d \frac{\partial c_i(x_t)}{\partial x_j}\,d\xi_t^j$ is an Itô stochastic differential.

In Section 3 we give conditions under which the solutions do not explode in finite time. For symmetric α-stable driving processes $\xi_t = \xi_t^{(\alpha)}$ we show in Section 4 that the solution process of the system (1) is always transient in dimensions $d \geqslant 3$. We consider it as an interesting open problem to find necessary and sufficient conditions for transience and recurrence for the system (1) in dimensions $d < 3$. Even in the case of a driving Wiener process (white noise) only some partial results are available for $d = 1$, see [4, 3].

2 Lévy Processes

The driving processes for our Newtonian system will be Lévy processes. Recall that a d-dimensional *Lévy process* $\{\xi_t\}_{t\geqslant 0}$ is a stochastic process with state space $\mathbb{R}^d$ and independent and stationary increments; its paths $t \mapsto \xi_t$ are continuous in probability which amounts to saying that there are almost surely no fixed discontinuities. We can (and will) always choose a modification with *càdlàg* (i.e., right-continuous with finite left limits) paths and $\xi_0 = 0$. Unless otherwise stated, we will always consider the augmented natural filtration of $\{\xi_t\}_{t\geqslant 0}$ which satisfies the "usual conditions". Because of the independent increment property the Fourier transform of the distribution of ξ_t is of the form

$$\mathbb{E}(e^{i\eta\xi_t}) = e^{-t\psi(\eta)}, \qquad t > 0,\ \eta \in \mathbb{R}^d,$$

with the *characteristic exponent* $\psi : \mathbb{R}^d \to \mathbb{C}$ which is given by the *Lévy-Khinchine formula*

$$\psi(\eta) = -i\beta\eta + \eta Q\eta + \int\limits_{\mathbb{R}^d\setminus\{0\}} \left(1 - e^{iy\eta} + i\,y\eta\,\mathbf{1}_{\{|y|<1\}}\right)\,\nu(dy). \tag{2}$$

Here $\beta \in \mathbb{R}^d$, $Q = (q_{ij}) \in \mathbb{R}^{d\times d}$ is a positive semidefinite matrix and ν is a Lévy measure, i.e., a Radon measure on $\mathbb{R}^d \setminus \{0\}$ with $\int_{y\neq 0} |y|^2 \wedge 1\,\nu(dy) < \infty$. The Lévy-triple (β, Q, ν) can also be used to obtain the *Lévy decomposition* of ξ_t,

$$\xi_t = W_t^Q + \iint\limits_{[0,t]\times\{0<|y|<1\}} y\,\tilde{N}(dy, ds) + \iint\limits_{[0,t]\times\{|y|\geqslant 1\}} y\,N(dy, ds) + \beta t \tag{3}$$

where $\Delta\xi_t := \xi_t - \xi_{t-}$, $\xi_{0-} := \xi_0$, $N(dy, ds) = \sum_{0\leqslant t\leqslant s} \mathbf{1}_{\{\Delta\xi_t\neq 0\}}\delta_{(\Delta\xi_t,t)}(dy, ds)$, is the canonical jump measure, $\tilde{N}(dy, ds) = N(dy, ds) - \nu(dy)\,ds$ is the compensated jump measure, W_t^Q is a Brownian motion with covariance matrix Q and βt is a deterministic drift with $\beta = \mathbb{E}\left(\xi_1 - \sum_{s\leqslant 1} \Delta\xi_s \mathbf{1}_{\{|\Delta\xi_s|\geqslant 1\}}\right)$. Notice that the first two terms in the above decomposition (3) are martingales.

Lemma 1. *Let $\{\xi_t\}_{t\geqslant 0}$ be a d-dimensional Lévy process whose jumps are bounded by $2R$. Then*

$$\mathbb{E}([\xi^i, \xi^j]_t) \leqslant t \max_{1\leqslant i,j\leqslant d} |q_{ij}| + t \int\limits_{0<|y|<2R} |y|^2\,\nu(dy), \qquad t > 0,$$

where $[\xi^i, \xi^j]_\bullet$ denotes the quadratic (co)variation process.

This Lemma is a simple consequence of the well-known formula

$$\mathbb{E}\big([\xi^i, \xi^j]_t\big) = \mathbb{E}\Bigg([W^i, W^j]_t + \sum_{s\leqslant t} \Delta\xi_s^i \Delta\xi_s^j\Bigg) = t\Bigg(q_{ij} + \int\limits_{|y|<2R} y^i y^j\,\nu(dy)\Bigg).$$

It is well-known that Lévy processes are Feller processes. The infinitesimal generator $(A, \mathfrak{D}(A))$ of the process (more precisely: of the associated Feller semigroup) is a *pseudo-differential operator* $A\big|_{C_c^\infty(\mathbb{R}^d)} = -\psi(D)$ with *symbol* $-\psi$, i.e.,

$$-\psi(D)u(x) := -(2\pi)^{-d/2} \int\limits_{\mathbb{R}^d} \psi(\eta)\widehat{u}(\eta)e^{iy\eta}\,d\eta, \qquad u \in C_c^\infty(\mathbb{R}^d), \tag{4}$$

where $\widehat{u}(\eta)$ denotes the Fourier transform of u. The test functions $C_c^\infty(\mathbb{R}^d)$ are an operator core. Later on, we will also use the following simple fact.

Lemma 2. *Let $u \in C_c^\infty(\mathbb{R}^d)$ and $u_R(x) := Ru(\frac{x}{R})$, $R \geqslant 1$. Then*

$$|\psi(D)u_R(x)| \leqslant C_\psi\, R \int\limits_{\mathbb{R}^d} (1+|\eta|^2)\, |\widehat{u}(\eta)|\, d\eta = C_{\psi,u}\, R$$

uniformly for all $x \in \mathbb{R}^d$ with an absolute constant $C_{\psi,u}$.

Proof. Observe that $\widehat{u}_R(\eta) = R^{d+1}\,\widehat{u}(R\eta)$. Therefore,

$$\begin{aligned}
|\psi(D)u_R(x)| &= (2\pi)^{-d/2} \left| \int\limits_{\mathbb{R}^d} e^{ix\eta}\psi(\eta)\,\widehat{u}_R(\eta)\, d\eta \right| \\
&\leqslant (2\pi)^{-d/2} R \int\limits_{\mathbb{R}^d} R^d\, |\psi(\eta)\,\widehat{u}(R\eta)|\; d\eta \\
&= (2\pi)^{-d/2} R \int\limits_{\mathbb{R}^d} \left|\psi\left(\tfrac{\eta}{R}\right)\widehat{u}(\eta)\right|\; d\eta \\
&\leqslant (2\pi)^{-d/2} C_\psi\, R \int\limits_{\mathbb{R}^d} \left(1 + \left|\tfrac{\eta}{R}\right|^2\right) |\widehat{u}(\eta)|\; d\eta \\
&\leqslant (2\pi)^{-d/2} C_\psi\, R \int\limits_{\mathbb{R}^d} \left(1 + |\eta|^2\right) |\widehat{u}(\eta)|\; d\eta,
\end{aligned}$$

where we used that $|\psi(\eta)| \leqslant C_\psi(1+|\eta|^2)$ for all $\eta \in \mathbb{R}^d$ with some absolute constant $C_\psi > 0$. Since $u \in C_c^\infty(\mathbb{R}^d)$, $\widehat{u}$ is a rapidly decreasing function which means that the integral in the last line is finite. ☐

Our standard references for the analytic theory of Lévy and Feller processes is the book [10] by Jacob, see also [11]; for stochastic calculus of semimartingales and stochastic differential equations we use Protter [30].

3 Non-explosion

Let $(X_t, P_t) = (X(t, x_0, p_0), P(t, x_0, p_0))$ be a solution of the system (1) with initial condition $(x_0, p_0) \in \mathbb{R}^{2d}$ at $t = 0$, where $\xi_t = (\xi_t^1, \dots, \xi_t^d)$ is a d-dimensional Lévy process, $d \geqslant 1$, $c \in C^2(\mathbb{R}^d, \mathbb{R}^d)$, $V \in C^2(\mathbb{R}^d)$, $V \geqslant 0$ and $\partial c/\partial x$ is uniformly bounded. Clearly, these conditions ensure local (i.e., for small times) existence and uniqueness of the solution, see e.g., [30].

The random times

$$T_m := \inf\{s \geqslant 0 : |X_s| \vee |P_s| \geqslant m\} \tag{5}$$

are stopping times w.r.t. the (augmented) natural filtration of the Lévy process $\{\xi_t\}_{t\geqslant 0}$ and so is the *explosion time* $T_\infty := \sup_m T_m$ of the system (1).

Theorem 3. *Under the assumptions stated above, the explosion time T_∞ of the system* (1) *is almost surely infinite, i.e.,* $\mathbb{P}(T_\infty = \infty) = 1$.

Proof. Step 1. Let $\tau_m := \inf\{s \geqslant 0 : |P_s| \geqslant m\}$ and $\tau_\infty := \sup_m \tau_m$. It is clear that $T_m \leqslant \tau_m$ and so $T_\infty \leqslant \tau_\infty$. Suppose that $T_\infty(\omega) < t < \tau_m(\omega) \leqslant \tau_\infty(\omega)$ for some $t > 0$ and $m \in \mathbb{N}$. From the first equation in (1) we deduce that for every $k \in \mathbb{N}$

$$\sup_{s\in[0,T_k(\omega)]} |X_s(\omega)| \leqslant |x_0| + t \sup_{s\in[0,t]} |P_s(\omega)| \leqslant |x_0| + tm.$$

On the other hand, since $T_k(\omega) < T_\infty(\omega) < t < \tau_\infty(\omega)$, we find that $\sup_{k\in\mathbb{N}} \sup_{s\in[0,T_k]} |X_s(\omega)| = \infty$. This, however, leads to a contradiction, and so $\tau_\infty = T_\infty$.

Step 2. We will show that $\mathbb{P}(\tau_\infty = \infty) = 1$. Set $H(x,p) := \frac{1}{2}p^2 + V(x)$ and $H_t = H(X_t, P_t)$. Since $H(x,p)$ is twice continuously differentiable, we can use Itô's formula (for jump processes and in the slightly unusual form of Protter [30, p. 71, (***)]). For this observe that only the quadratic variation of the Lévy process $[\xi,\xi] := ([\xi^i,\xi^j])_{ij} \in \mathbb{R}^{d\times d}$ contributes to the quadratic variation of $\{(X_t,P_t)\}_{t\geqslant 0}$:

$$[(X,P),(X,P)] = \begin{pmatrix} 0 & 0 \\ 0 & \left[\frac{\partial c}{\partial x}\,\xi, \frac{\partial c}{\partial x}\,\xi\right] \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \left(\frac{\partial c}{\partial x}\right)[\xi,\xi]\left(\frac{\partial c}{\partial x}\right)^T \end{pmatrix} \in \mathbb{R}^{2d\times 2d}.$$

Therefore,

$$dH_t = P_{t-}\,dP_t + \frac{1}{2}\operatorname{tr}\left(\frac{\partial c(X_{t-})}{\partial x}\,d[\xi,\xi]_t\left(\frac{\partial c(X_{t-})}{\partial x}\right)^T\right) + \frac{\partial V(X_t)}{\partial x}\,P_t\,dt + \Sigma_t,$$

where

$$\Sigma_t = \frac{1}{2}\sum_{0\leqslant s\leqslant t}\left(P_s^2 - P_{s-}^2 - 2P_{s-}(P_s - P_{s-}) - (P_s - P_{s-})^2\right) = 0.$$

The first equation in (1), $dX_t = P_t\,dt$, implies that X_t is a continuous function; the second equation, $dP_t = -\partial V(X_t)/\partial x\,dt - \partial c(X_t)/\partial x\,d\xi_t$, gives

$$dH_t = -P_{t-}\frac{\partial c(X_t)}{\partial x}\,d\xi_t + \frac{1}{2}\operatorname{tr}\left(\frac{\partial c(X_t)}{\partial x}d[\xi,\xi]_t\left(\frac{\partial c(X_t)}{\partial x}\right)^T\right). \tag{6}$$

Let $\sigma_R := \inf\{t > 0 : |\xi_t| \geqslant R\}$ be the first exit time of the process $\{\xi_t\}_{t\geqslant 0}$ from the ball $B_R(0)$. Then

$$\sigma = \sigma_{\ell,m,R} := \ell \wedge \sigma_R \wedge \tau_m, \qquad \ell, m \in \mathbb{N},$$

is again a stopping time and we calculate from (6) that

$$\begin{aligned} H_{\sigma-} - H_0 &= -\int_0^{\sigma-} P_{t-}\frac{\partial c(X_t)}{\partial x}\,d\xi_t + \frac{1}{2}\int_0^{\sigma-}\operatorname{tr}\left(\frac{\partial c(X_t)}{\partial x}\,d[\xi,\xi]_t\left(\frac{\partial c(X_t)}{\partial x}\right)^T\right) \\ &= \mathbf{I} + \mathbf{II}. \end{aligned} \tag{7}$$

Step 3. Recall that $-\psi(D)$ is the generator of the Lévy process ξ_t. We want to estimate $|\mathbb{E}(\mathbf{I})|$. For this purpose choose a function $\phi \in C_c^\infty(\mathbb{R}^d,\mathbb{R}^d)$ such that $\phi(x) = x$ if $|x| \leqslant 1$, $\operatorname{supp}\phi \subset \{x : |x| \leqslant 2\}$ and define $\phi_R(x) = R\phi\left(\frac{x}{R}\right)$. Clearly,

$$\phi_R(\xi_t) = \xi_t, \qquad t < \sigma_R, \tag{8}$$

and, since $\phi_R \in C_c^\infty(\mathbb{R}^d) \subset \mathfrak{D}(A)$ is in the domain of the generator of ξ_t, we find that

$$M_t^{\phi_R} := \phi_R(\xi_t) + \int_0^t \psi(D)\phi_R(\xi_s)ds \tag{9}$$

is an L^2-martingale (w.r.t. the natural filtration of $\{\xi_t\}_{t\geqslant 0}$). The stopped process $(M^{\phi_R}_{t\wedge\tau_m\wedge\ell})_{t\geqslant 0}$ is again an L^2-martingale for fixed $m, \ell \in \mathbb{N}$. We can now use (8) and (9) to get

$$\mathbf{I} = -\int_0^{\sigma-} P_{t-}\frac{\partial c(X_t)}{\partial x}\, dM^{\phi_R}_{t\wedge\tau_m\wedge\ell} + \int_0^{\sigma-} P_{t-}\frac{\partial c(X_t)}{\partial x}\, \psi(D)\phi_R(\xi_t)\, dt = \mathbf{I}' + \mathbf{I}''.$$

Clearly, $\int_0^\bullet P_{t-}(\partial c(X_t)/\partial x)\, dM^{\phi_R}_{t\wedge\tau_m\wedge\ell}$ is a local martingale. Since

$$\begin{aligned}\Big[\int_0^\bullet P_{s-}\frac{\partial c(X_s)}{\partial x}\, dM^{\phi_R}_{s\wedge\tau_m\wedge\ell}, \int_0^\bullet P_{s-}\frac{\partial c(X_s)}{\partial x}\, dM^{\phi_R}_{s\wedge\tau_m\wedge\ell}\Big]_t \\ = \int_0^t P^2_{s-}\left(\frac{\partial c(X_s)}{\partial x}\right)^2 d[M^{\phi_R}_\bullet, M^{\phi_R}_\bullet]_{s\wedge\tau_m\wedge\ell} \\ = \int_0^{t\wedge\tau_m\wedge\ell} P^2_{s-}\left(\frac{\partial c(X_s)}{\partial x}\right)^2 d[M^{\phi_R}_\bullet, M^{\phi_R}_\bullet]_{s\wedge\tau_m\wedge\ell}\end{aligned}$$

we find for every $t > 0$

$$\begin{aligned}\left|\mathbb{E}\Big[\int_0^\bullet P_{s-}\frac{\partial c(X_s)}{\partial x}\, dM^{\phi_R}_{s\wedge\tau_m\wedge\ell}, \int_0^\bullet P_{s-}\frac{\partial c(X_s)}{\partial x}\, dM^{\phi_R}_{s\wedge\tau_m\wedge\ell}\Big]_t\right| \\ \leqslant m^2 \left\|\frac{\partial c}{\partial x}\right\|^2_\infty \mathbb{E}\, [M^{\phi_R}_\bullet, M^{\phi_R}_\bullet]_t < \infty,\end{aligned}$$

where we used that $|P_{s-}| \leqslant m$ if $s \leqslant \ell\wedge\tau_m$ and that $M_t^{\phi_R}$ is an L^2-martingale. This shows that $\int_0^\bullet P_{t-}(\partial c(X_t)/\partial x)\, dM_t^{\phi_R}$ is a martingale (cf. [30], p.66 Corollary 3) and we may apply optional stopping to the bounded stopping time σ to get

$$\begin{aligned}\mathbb{E}(\mathbf{I}') &= -\mathbb{E}\left(\int_0^\sigma P_{t-}\, \frac{\partial c(X_t)}{\partial x}\, dM_t^{\phi_R}\right) + \mathbb{E}\left(P_{\sigma-}\frac{\partial c(X_\sigma)}{\partial x}\, \Delta M_\sigma^{\phi_R}\right) \\ &= \mathbb{E}\left(P_{\sigma-}\frac{\partial c(X_\sigma)}{\partial x}\, \Delta M_\sigma^{\phi_R}\right).\end{aligned}$$

Therefore

$$\left|\mathbb{E}(\mathbf{I}')\right| \leqslant md^2 \left\|\frac{\partial c}{\partial x}\right\|_\infty \mathbb{E}\left|\Delta M_\sigma^{\phi_R}\right| \leqslant 2mRd^2\left\|\frac{\partial c}{\partial x}\right\|_\infty \|\phi\|_\infty, \tag{10}$$

where we used

$$\left|\Delta M_\sigma^{\phi_R}\right| = |\phi_R(\xi_\sigma) - \phi_R(\xi_{\sigma-})| \leqslant 2R\|\phi\|_\infty$$

and the notation

$$\left\|\frac{\partial c}{\partial x}\right\|_\infty := \max_{i,j=1,\dots,d} \sup_{x\in\mathbb{R}^d} \left|\frac{\partial c_i(x)}{\partial x_j}\right|.$$

Step 4. For the estimate of $\mathbb{E}(\mathbf{I}'')$, we use Lemma 2 with $u = \phi$ to get $\|\psi(D)\phi_R\|_\infty \leqslant C_{\psi,\phi}$, and also $\sigma \leqslant \ell$, so

$$\left|\mathbb{E}(\mathbf{I}'')\right| \leqslant C_{\psi,\phi}\, R\, \mathbb{E}\left(\sup_{t<\sigma}\left|P_{t-}\frac{\partial c(X_t)}{\partial x}\right|\right)\ell \leqslant C_2 \left\|\frac{\partial c}{\partial x}\right\|_\infty Rm\,\ell. \tag{11}$$

Put together, the estimates (10), (11) give

$$|\mathbb{E}(\mathbf{I})| \leqslant C_3 Rm\,\ell. \tag{12}$$

Step 5. We proceed with $|\mathbb{E}(\mathbf{II})|$. From

$$\|AB\|_\infty \leqslant d\|A\|_\infty\|B\|_\infty, \qquad A, B \in \mathbb{R}^{d\times d},$$

where $\|A\|_\infty = \max_{i,j=1,\dots,d}|A_{ij}|$, we get

$$\int_0^t \operatorname{tr}\left[\frac{\partial c(X_s)}{\partial x}\, d[\xi,\xi]_s \left(\frac{\partial c(X_s)}{\partial x}\right)^T\right] \leqslant d^3 \left\|\frac{\partial c}{\partial x}\right\|_\infty^2 \|[\xi,\xi]_t\|_\infty.$$

Since we have $\sup_{s\leqslant t}|\xi_s| \leqslant R$ for $t < \sigma_R$, the jumps $|\Delta\xi_s|$, $s \leqslant t$, cannot exceed $2R$. Lemma 1 then shows

$$\mathbb{E}\left([\xi^i,\xi^j]_{\ell\wedge\sigma_R-}\right) \leqslant \ell \int_{0<|y|\leqslant 2R} |y|^2\,\nu(dy) + \ell\,\|Q\|_\infty$$

and so

$$|\mathbb{E}(\mathbf{II})| \leqslant C_4\,\ell\left(\int_{0<|y|\leqslant 2R} |y|^2\,\nu(dy) + \|Q\|_\infty\right). \tag{13}$$

Step 6. Combining (7), (12), (13) we obtain

$$\mathbb{E}\left(H_{\sigma-}\right) \leqslant H_0 + C_3 Rm\,\ell + C_4\,\ell\left(\int_{0<|y|\leqslant 2R} |y|^2\nu(dy) + \|Q\|_\infty\right). \tag{14}$$

On the other hand, by Jensen's inequality,

$$\begin{aligned}\mathbb{E}\left(H_{\sigma-}\right) = \frac{1}{2}\,\mathbb{E}\left(P_{\sigma-}^2\right) + \mathbb{E}(V(X_{\sigma-})) &\geqslant \frac{1}{2}\,\mathbb{E}\left(P_{\sigma-}^2\right)\\ &\geqslant \frac{1}{2}\left[\mathbb{E}\left(|P_{\sigma-}|\right)\right]^2\\ &\geqslant \frac{1}{2}\left[\mathbb{E}\left(|P_{\ell\wedge\tau_m\wedge\sigma_R-}|\,\mathbf{1}_{\{\tau_m<\ell\wedge\sigma_R\}}\right)\right]^2\\ &= \frac{1}{2}\left[\mathbb{E}\left(|P_{\tau_m} - \Delta P_{\tau_m}|\,\mathbf{1}_{\{\tau_m<\ell\wedge\sigma_R\}}\right)\right]^2.\end{aligned}$$

Clearly, $|P_{\tau_m}| \geqslant m$ and, since on $\{s < \sigma_R\}$ the driving Lévy process has jumps of size $|\Delta\xi_s| \leqslant 2R$, we find from (1) that

$$|\Delta P_{\tau_m}|\,\mathbf{1}_{\{\tau_m<\ell\wedge\sigma_R\}} \leqslant 2R\left\|\frac{\partial c}{\partial x}\right\|_\infty \mathbf{1}_{\{\tau_m<\ell\wedge\sigma_R\}}.$$

Choosing m sufficiently large, say $m > 2R\|(\partial c/\partial x)\|_\infty$, we arrive at

$$\begin{aligned}\mathbb{E}\left(H_{\sigma-}\right) &\geqslant \frac{1}{2}\left[\mathbb{E}\left(m-|\Delta P_{\tau_m}|\right)\,\mathbf{1}_{\{\tau_m<\ell\wedge\sigma_R\}}\right]^2\\ &\geqslant \frac{1}{2}\Big(m-2R\left\|\frac{\partial c}{\partial x}\right\|_\infty\Big)^2\left\{\mathbb{P}\left(\tau_m<\ell\wedge\sigma_R\right)\right\}^2. \end{aligned} \tag{15}$$

We can now combine (14) and (15) to find

$$\begin{aligned}\left\{\mathbb{P}\left(\tau_m<\ell\wedge\sigma_R\right)\right\}^2 \leqslant &\frac{2(H_0+C_3Rm\ell)}{(m-2R\|(\partial c/\partial x)\|_\infty)^2}\\ &+\frac{2C_4\ell}{(m-2R\|(\partial c/\partial x)\|_\infty)^2}\Big(\int\limits_{0<|y|\leqslant 2R}|y|^2\nu(dy)+\|Q\|_\infty\Big).\end{aligned}$$

Letting first $m\to\infty$ and then $R\to\infty$ shows $\mathbb{P}(\tau_\infty\leqslant\ell)=0$ for all $\ell\in\mathbb{N}$, so $\mathbb{P}(\tau_\infty=\infty)=1$, and the claim follows. $\square$

4 Transience

We will now prove that the solution $\{(X_t,P_t)\}_{t\geqslant 0}$ of the Newton system (1) is transient, at least if the driving noise is a symmetric stable Lévy process $\xi_t=\xi_t^{(\alpha)}$ with index $\alpha\in(0,2)$. Symmetric α-stable Lévy processes have no drift, no Brownian part and their Lévy measures are $\nu(dy)=c_\alpha\,|y|^{-d-\alpha}\,dy$, where

$$c_\alpha=\frac{\alpha\,2^{\alpha-1}\,\Gamma\left(\frac{\alpha+d}{2}\right)}{\pi^{\alpha/2}\Gamma\left(1-\frac{\alpha}{2}\right)}. \tag{16}$$

We restrict ourselves to presenting this particular case, but it is clear that, with minor alterations, the proof of Theorem 6 below remains valid for *any* driving Lévy process *with rotationally symmetric Lévy measure.*

Our proof is be based on the following result which extends a well-known transience criterion for diffusion processes to jump processes, see for instance [8] or [21].

Denote by $\{T_t\}_{t\geqslant 0}$ the operator semigroup associated with a stochastic process and let $(A,\mathfrak{D}(A))$ be its generator. The *full generator* is the set

$$\widehat{A}:=\left\{(f,g)\in C_b\times C_b\;:\;T_tf-f=\int_0^t T_sg\,ds\right\},$$

see Ethier, Kurtz [7] p. 24. It is clear that $(u,Au)\in\widehat{A}$ for all $u\in\mathfrak{D}(A)$.

Lemma 4. *Let $\{\eta_t\}_{t\geqslant 0}$ be an $\mathbb{R}^n$-valued, càdlàg strong Markov process with generator $(A, \mathfrak{D}(A))$ and full generator $\widehat{A}$. Let $D \subset \mathbb{R}^n$ be a bounded Borel set and assume that there exists a sequence $\{u_k\}_{k\in\mathbb{N}} \subset C_b(\mathbb{R}^n)$ and some function $u \in C(\mathbb{R}^n)$, such that the following conditions are satisfied:*

(i) *A has an extension $\tilde{A}$ such that $\tilde{A}u_k$ is pointwise defined, $(u_k, \tilde{A}u_k) \in \widehat{A}$ and $\lim_{k\to\infty}(u_k, \tilde{A}u_k) = (u, \tilde{A}u)$ exists locally uniformly.*

(ii) *$u \geqslant 0$ and $\inf\limits_D u > a > 0$ for some $a > 0$.*

(iii) *$u(y_0) < a$ for some $y_0 \notin \overline{D}$.*

(iv) *$\tilde{A}u \leqslant 0$ in D^c.*

Then $\{\eta_t\}_{t\geqslant 0}$ is transient.

Proof. Since $(u_k, \tilde{A}u_k) \in \widehat{A}$, we know that

$$M_t^k = u_k(\eta_t) - \int_0^t \tilde{A}u_k(\eta_s)\,ds, \qquad k \in \mathbb{N},$$

are martingales, see Ethier, Kurtz [7, p. 162, Prop. 4.1.7]. We set

$$\tau_D = \inf\{t > 0 \,:\, \eta_t \in D\} \quad \text{and} \quad \sigma_R = \inf\{t > 0 \,:\, |\eta_t - \eta_0| > R\}$$

and from an optional stopping argument we find for any fixed $T > 0$

$$\mathbb{E}^{y_0}\left(M^k_{\tau_D\wedge\sigma_R\wedge T}\right) = \mathbb{E}^{y_0}(M_0^k) = \mathbb{E}^{y_0}(u_k(\eta_0)).$$

On the other hand,

$$\mathbb{E}^{y_0}\left(M^k_{\tau_D\wedge\sigma_R\wedge T}\right) = \mathbb{E}^{y_0}\left(u_k(\eta_{\tau_D\wedge\sigma_R\wedge T}) - \int_0^{\tau_D\wedge\sigma_R\wedge T} \tilde{A}u_k(\eta_s)ds\right),$$

and because of assumption (i) we can pass to the limit $k \to \infty$ to get

$$\begin{aligned}
a > u(y_0) &= \lim_{k\to\infty} u_k(y_0) \\
&= \lim_{k\to\infty} \mathbb{E}^{y_0}\left(u_k(\eta_{\tau_D\wedge\sigma_R\wedge T}) - \int_0^{\tau_D\wedge\sigma_R\wedge T} \tilde{A}u_k(\eta_s)ds\right) \\
&= \mathbb{E}^{y_0}\left(u(\eta_{\tau_D\wedge\sigma_R\wedge T}) - \int_0^{\tau_D\wedge\sigma_R\wedge T} \tilde{A}u(\eta_s)ds\right) \\
&\geqslant \mathbb{E}^{y_0}\big(u(\eta_{\tau_D\wedge\sigma_R\wedge T})\big) \\
&\geqslant \mathbb{E}^{y_0}\left(u(\eta_{\tau_D\wedge\sigma_R\wedge T})\,\mathbf{1}_{\{\tau_D<\infty\}}\right),
\end{aligned}$$

where we used in the penultimate step that $\tilde{A}u\big|_{D^c} \leqslant 0$.

As $u \in C^+(\mathbb{R}^n)$, we may use dominated convergence and let $T \to \infty$ and Fatou's Lemma to let $R \to \infty$. Thus,

$$
\begin{aligned}
a > u(y_0) \geqslant \liminf_{R\to\infty} \mathbb{E}^{y_0}\left(u(\eta_{\tau_D\wedge\sigma_R})\mathbf{1}_{\{\tau_D<\infty\}}\right) \geqslant \mathbb{E}^{y_0}\left(u(\eta_{\tau_D})\mathbf{1}_{\{\tau_D<\infty\}}\right) \\
\geqslant (\inf_D u)\,\mathbb{P}^{y_0}(\tau_D<\infty) > a\,\mathbb{P}^{y_0}(\tau_D<\infty).
\end{aligned}
$$

Therefore, $\mathbb{P}^{y_0}(\tau_D < \infty) < 1$, and, see e.g [2], $\{\eta_t\}_{t\geqslant 0}$ is transient. $\square$

We will now turn to the task to determine the infinitesimal generator of the solution process $\{(X_t, P_t)\}_{t\geqslant 0}$. The following result is, in various settings, common knowledge. We could not find a precise reference in our situation, though. Since we need some technical details of the proof, we include the standard argument.

Lemma 5. *Let $\{\xi_t\}_{t\geqslant 0}$ be a d-dimensional Lévy process with characteristic exponent ψ and Lévy triple (α, Q, ν). The (pointwise) infinitesimal generator of the process $(X_t, P_t) = (X(t, x_0, p_0), P(t, x_0, p_0))$ solving (1) is of the form*

$$
\begin{aligned}
Au(x,p) = {} & \frac{\partial u(x,p)}{\partial x}\,p - \frac{\partial u(x,p)}{\partial p}\left(\frac{\partial V(x)}{\partial x} + \frac{\partial c(x)}{\partial x}\,\beta\right) \\
& + \frac{1}{2}\mathrm{tr}\left(\frac{\partial^2 u(x,p)}{\partial p^2}\left(\frac{\partial c(x)}{\partial x}\right) Q \left(\frac{\partial c(x)}{\partial x}\right)^T\right) \\
& + \int\limits_{\mathbb{R}^d\setminus\{0\}} \left(u(x, p - \tfrac{\partial c(x)}{\partial x}\,y) - u(x,p) + \frac{\partial u(x,p)}{\partial p}\frac{\partial c(x)}{\partial x}\,y\,\mathbf{1}_{\{|y|<1\}}\right)\nu(dy).
\end{aligned}
$$

for all $u \in C_c^2(\mathbb{R}^d \times \mathbb{R}^d)$ and with $\beta = \mathbb{E}^0\left(\xi_1 - \sum_{0\leqslant s\leqslant 1} \Delta\xi_\tau\,\mathbf{1}_{\{|\Delta\xi_s|\geqslant 1\}}\right)$. In particular, the pairs (u, Au), $u \in C_c^2(\mathbb{R}^d \times \mathbb{R}^d)$, are in the full generator $\widehat{A}$ of the process.

Proof. For $u = u(x,p) \in C_c^2(\mathbb{R}^d \times \mathbb{R}^d)$ we can use Itô's formula (for jump processes, now in the usual form [30, p. 70, Theorem II.32]) and get with a similar calculation to the one made in the proof of Theorem 3

$$
\begin{aligned}
u(X_t,P_t) - u(x_0,p_0) = {} & \int_0^t \frac{\partial u}{\partial x}\,P_s\,ds - \int_0^t \frac{\partial u}{\partial p}\frac{\partial V}{\partial x}\,ds - \int_0^t \frac{\partial u}{\partial p}\frac{\partial c}{\partial x}\,d\xi_s \\
& + \frac{1}{2}\int_0^t \mathrm{tr}\left(\frac{\partial^2 u}{\partial p^2}\left(\frac{\partial c}{\partial x}\right) Q \left(\frac{\partial c}{\partial x}\right)^T\right) ds \\
& + \sum_{0\leqslant s\leqslant t}\left(u(X_s,P_s) - u(X_s,P_{s-}) + \frac{\partial u}{\partial p}(X_s,P_{s-})\frac{\partial c}{\partial x}\,\Delta\xi_s\right).
\end{aligned}
$$

Here we used the fact that the continuous martingale part of ξ_t is W_t^Q, and so $[\xi,\xi]_t^c = [W^Q, W^Q]_t = Qt$. Note that we suppressed arguments in those places where no ambiguity is possible. Since $P_s = P_{s-} + \Delta P_s = P_{s-} - \frac{\partial c}{\partial x}\,\Delta\xi_s$ we find, using the Lévy decomposition (3),

$$
\begin{aligned}
&u(X_t,P_t) - u(x_0,p_0)\\
&= \int_0^t \frac{\partial u}{\partial x}\,P_s\,ds - \int_0^t \frac{\partial u}{\partial p}\frac{\partial V}{\partial x}\,ds - \int_0^t \frac{\partial u}{\partial p}\frac{\partial c}{\partial x}\,\beta\,ds - \int_0^t \frac{\partial u}{\partial p}\frac{\partial c}{\partial x}\,dW_s^Q\\
&\quad - \int_0^t \frac{\partial u}{\partial p}\frac{\partial c}{\partial x}\int_{0<|y|<1} y\,\tilde N(dy,ds) + \frac12\int_0^t \operatorname{tr}\left(\frac{\partial^2 u}{\partial p^2}\left(\frac{\partial c}{\partial x}\right)Q\left(\frac{\partial c}{\partial x}\right)^T\right)ds\\
&\quad + \iint\Big(u(X_s,P_{s-} - \tfrac{\partial c}{\partial x}\,y) - u(X_s,P_{s-}) + \tfrac{\partial u(X_s,P_{s-})}{\partial p}\tfrac{\partial c}{\partial x}\,y\,\mathbf{1}_{\{|y|<1\}}\Big)\,\tilde N(dy,ds)\\
&\quad + \iint\Big(u(X_s,P_{s-} - \tfrac{\partial c}{\partial x}\,y) - u(X_s,P_{s-}) + \tfrac{\partial u(X_s,P_{s-})}{\partial p}\tfrac{\partial c}{\partial x}\,y\,\mathbf{1}_{\{|y|<1\}}\Big)\,\nu(dy)\,ds
\end{aligned}
$$

with the double integrals ranging over $[0,t]\times\mathbb{R}^d\setminus\{0\}$. The function u has compact support, and we may take expectations on both sides of the above relation and differentiate in t. Since the terms driven by $\tilde N(dy,ds)$ or dW_s^Q are martingales, we find

$$
\begin{aligned}
&\frac{d}{dt}\mathbb{E}\left(u(X_t,P_t)\right)\Big|_{t=0}\\
&= \frac{\partial u(x_0,p_0)}{\partial x}\,p_0 - \frac{\partial u(x_0,p_0)}{\partial p}\frac{\partial V(x_0)}{\partial x} - \frac{\partial u(x_0,p_0)}{\partial p}\frac{\partial c(x_0)}{\partial x}\,\beta\\
&\quad + \frac12\operatorname{tr}\left(\frac{\partial^2 u(x_0,p_0)}{\partial p^2}\left(\frac{\partial c(x_0)}{\partial x}\right)Q\left(\frac{\partial c(x_0)}{\partial x}\right)^T\right)\\
&\quad + \int_{\mathbb{R}^d\setminus\{0\}}\Big(u(x_0,p_0 - \tfrac{\partial c(x_0)}{\partial x}\,y) - u(x_0,p_0) + \tfrac{\partial u(x_0,p_0)}{\partial p}\tfrac{\partial c(x_0)}{\partial x}\,y\,\mathbf{1}_{\{|y|<1\}}\Big)\,\nu(dy),
\end{aligned}
$$

which is what we claimed. Notice, that the convergence is pointwise, so that it is not clear that $C_c^2(\mathbb{R}^d\times\mathbb{R}^d)$ is in the domain of the generator. However, our calculation shows that $Au\in C_b(\mathbb{R}^d\times\mathbb{R}^d)$ and

$$
\mathbb{E}\,u(X_t,P_t) - u(x_0,p_0) = \int_0^t \mathbb{E}\,(Au)(X_s,P_s)\,ds
$$

which means that (u,Au) is in the full generator $\widehat{A}$. □

If the driving Lévy process has no drift, no Brownian part and a rotationally symmetric Lévy measure, the form of the infinitesimal generator becomes much simpler. In this case we have for all $u\in C_c^2(\mathbb{R}^d\times\mathbb{R}^d)$

$$
\begin{aligned}
Au(x,p) &= \frac{\partial u(x,p)}{\partial x}\,p - \frac{\partial u(x,p)}{\partial p}\frac{\partial V(x)}{\partial x}\\
&\quad + \mathrm{v.p.}\int_{\mathbb{R}^d}\Big(u(x,p - \tfrac{\partial c(x)}{\partial x}\,y) - u(x,p)\Big)\,\nu(dy),
\end{aligned}
\tag{17}
$$

where v.p.$\int_{\mathbb{R}^d} f(y)\,\nu(dy) := \lim_{\varepsilon\to 0}\int_{|y|>\varepsilon} f(y)\,\nu(dy)$ stands for the principal value integral. It is not hard to see that

$$
\begin{aligned}
&\text{v.p.}\int_{\mathbb{R}^d} \Big(u(x,p-\tfrac{\partial c(x)}{\partial x}\,y)-u(x,p)\Big)\,\nu(dy)\\
&\qquad = \int_{\mathbb{R}^d\setminus\{0\}} \Big(u(x,p-\tfrac{\partial c(x)}{\partial x}\,y)-u(x,p)+\tfrac{\partial u(x,y)}{\partial x}\tfrac{\partial c(x)}{\partial x}\,y\,\mathbf{1}_{\{|y|<1\}}\Big)\,\nu(dy)
\end{aligned}
$$

or also

$$
= \frac{1}{2}\int_{\mathbb{R}^d\setminus\{0\}} \Big(u(x,p-\tfrac{\partial c(x)}{\partial x}\,y)+u(x,p+\tfrac{\partial c(x)}{\partial x}\,y)-2u(x,p)\Big)\,\nu(dy)
$$

holds. The latter two representations do exist in the sense of ordinary integrals (just use a simple Taylor expansion for u up to order two) and are frequently used in the literature. For our purposes, formula (17) is better suited. Notice that all three representations extend A onto C^2.

Theorem 6. *Let $d\geqslant 3$, $V\in C^2(\mathbb{R}^d)$, $c\in C^2(\mathbb{R}^d,\mathbb{R}^d)$ and $\{\xi_t\}_{t\geqslant 0}$ be a symmetric α-stable Lévy process, $0<\alpha<2$. Then the process $\{(X_t,P_t)\}_{t\geqslant 0}$ solving* (1) *is transient.*

Proof. We want to apply Lemma 4. Take the function

$$
u_\gamma(x,p) = (H(x,p)-V_0)^{-\gamma} = \big(\tfrac{1}{2}p^2+V(x)-V_0\big)^{-\gamma}
$$

with $V_0=\inf V-1$ and with a parameter $\gamma>0$ which we will choose later. It is not hard to see that for this $u=u_\gamma(x,p)$ and

$$
D := \Big\{(x,p)\in\mathbb{R}^{2d}\;:\;|x|+|p|\leqslant 1\Big\}, \qquad a := \tfrac{1}{2}\min_{(x,p)\in D} u_\gamma(x,p)
$$

conditions (ii), (iii) of Lemma 4 are satisfied.

Moreover, we have

$$
\frac{\partial u_\gamma}{\partial x}\,p - \frac{\partial u_\gamma}{\partial p}\,\frac{\partial V}{\partial x} = 0.
$$

Since $\{\xi_t\}_{t\geqslant 0}$ is a symmetric α-stable process, its Lévy measure is of the form $\nu(dy) = c_\alpha\,|y|^{-d-\alpha}\,dy$ with c_α given by (16), and (17) shows that

$$
\tilde{A}u_\gamma(x,p) = c_\alpha\,\text{v.p.}\int_{\mathbb{R}^d}\big(u_\gamma\big(x,p+\tfrac{\partial c}{\partial x}\,y\big)-u_\gamma(x,p)\big)\,\frac{dy}{|y|^{d+\alpha}}.
$$

We will see in Corollary 9 below (with $B=\partial c/\partial x$ and $b=2(V(x)-V_0)$) that we can choose $\gamma>0$ in such a way that $\tilde{A}u_\gamma(x,p)\leqslant 0$. This, however, means that also condition (iv) of Lemma 4 is met.

Let $\chi_k\in C_c^\infty(\mathbb{R}^d)$ be a cut-off function with $\mathbf{1}_{B_k(0)}\leqslant\chi_k\leqslant\mathbf{1}_{B_{2k}(0)}$ and set $u_k(x,p):=u_\gamma(x,p)\chi_k(x)\chi_k(p)$. Clearly, $u_k\in C_c^2(\mathbb{R}^d\times\mathbb{R}^d)$ and we know from Lemma 5 that the pair

(u_k, Au_k) is in the full generator $\widehat{A}$. The following considerations are close to those in [31]. Write $\|g\|_A = \|g\mathbf{1}_A\|_\infty$. Using a Taylor expansion we find for some $0 < \theta < 1$ and all $f \in C^2(\mathbb{R}^d \times \mathbb{R}^d)$

$$
\begin{aligned}
&f\big(x, p + \tfrac{\partial c}{\partial x} y\big) - f(x,p) \\
&\qquad = \frac{\partial f(x,p)}{\partial p}\,\frac{\partial c(x)}{\partial x}\,y + \frac{1}{2}\sum_{i,j=1}^{d} \frac{\partial^2 f(x, p + \theta \frac{\partial c}{\partial x}\, y)}{\partial p_i \partial p_j}\,\left(\frac{\partial c}{\partial x} y\right)_i \left(\frac{\partial c}{\partial x} y\right)_j
\end{aligned}
$$

and, therefore, for all compact sets $K \subset \mathbb{R}^d$ and $(x,p) \in K \times K$,

$$
\begin{aligned}
&\left| \text{v.p.}\int_{\mathbb{R}^d} \left(f\left(x, p + \tfrac{\partial c}{\partial x} y\right) - f(x,p)\right)\,\nu(dy)\right| \\
&\qquad \leqslant \left| \text{v.p.}\int_{|y|<1} \left(f\left(x, p + \tfrac{\partial c}{\partial x} y\right) - f(x,p)\right)\,\nu(dy)\right| + 2\int_{|y|\geqslant 1} \nu(dy)\,\|f\|_{K\times\mathbb{R}^d} \\
&\qquad \leqslant \frac{d^4}{2}\left\|\frac{\partial c}{\partial x}\right\|_K^2 \int_{0<|y|<1} |y|^2\,\nu(dy)\,\left\|\frac{\partial^2 f}{\partial p^2}\right\|_{K\times\tilde{K}} + 2\int_{|y|\geqslant 1} \nu(dy)\,\|f\|_{K\times\mathbb{R}^d},
\end{aligned}
$$

where $\tilde{K} = K + \{p \in \mathbb{R}^d \ : \ |p| \leqslant \|\partial c/\partial x\|_K\}$. Since the estimate of the local part in (17) is obvious, we find

$$
\|\tilde{A} f\|_{K\times K} \leqslant C\left(\|f\|_{K\times\mathbb{R}^d} + \left\|\frac{\partial f}{\partial x}\right\|_{K\times K} + \left\|\frac{\partial f}{\partial p}\right\|_{K\times K} + \left\|\frac{\partial^2 f}{\partial p^2}\right\|_{K\times\tilde{K}}\right),
$$

for any $f \in C^2(\mathbb{R}^d \times \mathbb{R}^d)$ with $\|f\|_{K\times\mathbb{R}^d} < \infty$ and with an absolute constant $C = C(K, c, V)$ depending only on K, $\|\partial c/\partial x\|_K$ and $\|\partial V/\partial x\|_K$. Since $p \mapsto u_\gamma(x,p)$ vanishes at infinity, condition (i) of Lemma 4 is satisfied for the sequence $(u_k, Au_k) \to (u_\gamma, \tilde{A}u_\gamma)$.

The theorem follows now directly from Lemma 4. □

Appendix

We will now give the somewhat technical proof that for some $\gamma > 0$ the function $u_\gamma(x,p) = \left(\frac{1}{2}p^2 + V(x) - V_0\right)^{-\gamma}$ which we used in the proof of Theorem 6 satisfies condition (iv) of Lemma 4. We begin with a few elementary lemmas.

Recall that Euler's Beta function $B(x,y)$ is given by

$$
B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt, \qquad x, y > 0, \tag{18}
$$

and satisfies the relations

$$
B(x,y) = B(y,x) \qquad \text{and} \qquad B(x,y) = \frac{x+y}{y}\,B(x, y+1), \tag{19}
$$

cf. Gradshteyn and Ryzhik [9, §8.38]. A change of variable in (18) according to $t = s^2$ yields

$$B(x,y) = \int_{-1}^{1} (s^2)^{x-\frac{1}{2}}(1-s^2)^{y-1}\,ds, \qquad x,y>0.$$

Lemma 7. *For any $v \in \mathbb{R}\setminus\{0\}$, $a \geqslant 1$, $d \geqslant 3$ we have*

$$J(v) = \int_{-1}^{1} (1-s^2)^{\frac{d-3}{2}} \ln(v^2+2vs+a)\,ds > \ln(a)\, I_{\frac{d-3}{2}}. \tag{20}$$

Proof. We observe that $J(v) = J(-v)$ and

$$\ln(v^2+2vs+a) - \ln(a) = \ln\left(\frac{v^2}{a} + 2\frac{v}{a}s + 1\right) \geqslant \ln\left(\frac{v^2}{a^2} + 2\frac{v}{a}s + 1\right).$$

Therefore, we may assume that $a = 1$ and $v \geqslant 0$. Since $J(0) = \ln(a) = 0$, it is enough to show that $J(v)$ is increasing. This is clear for $v \geqslant 1$ since $v \mapsto v^2 + 2vs + 1$ increases for all parameter values $|s| \leqslant 1$. For $0 < v < 1$ we calculate the derivative

$$J'(v) = 2\int_{-1}^{1} \frac{v+s}{v^2+2vs+1}\,(1-s^2)^{\frac{d-3}{2}}\,ds.$$

In the case $d = 3$ a few lines of simple calculations give

$$J'(v) = \left(1 - \frac{1}{v^2}\right)\ln\left(\frac{1+v}{1-v}\right) + \frac{2}{v}$$

which is clearly positive. If $d > 3$, we use the symmetry of the measure $(1-s^2)^{\frac{d-3}{2}}\,ds$ and find

$$\begin{aligned}
J'(v) &= \int_{-1}^{1} \left(\frac{v+s}{v^2+2vs+1} + \frac{v-s}{v^2-2vs+1}\right)(1-s^2)^{\frac{d-3}{2}}\,ds \\
&= 2v\int_{-1}^{1} \frac{v^2+1-2s^2}{(v^2+1)^2-4v^2s^2}\,(1-s^2)^{\frac{d-3}{2}}\,ds \\
&= \frac{2v}{(v^2+1)^2}\int_{-1}^{1} (v^2+1-2s^2)\sum_{j=0}^{\infty}\left(\frac{2v}{v^2+1}\right)^{2j} s^{2j}\,(1-s^2)^{\frac{d-3}{2}}\,ds,
\end{aligned}$$

since $2v(v^2+1)^{-1} \leqslant 1$. The integrand can be written as

$$\begin{aligned}
&(v^2+1-2s^2)\sum_{j=0}^{\infty}\left(\frac{2v}{v^2+1}\right)^{2j} s^{2j} \\
&= (v^2+1)\sum_{j=0}^{\infty}\left(\frac{2v}{v^2+1}\right)^{2j} s^{2j} - 2\sum_{j=0}^{\infty}\left(\frac{2v}{v^2+1}\right)^{2j} s^{2j+2}
\end{aligned}$$

$$
\begin{aligned}
&= (v^2+1) + \sum_{j=1}^{\infty} \left\{ (v^2+1) \left(\frac{2v}{v^2+1} \right)^{2j} - 2 \left(\frac{2v}{v^2+1} \right)^{2j-2} \right\} s^{2j} \\
&= (v^2+1) + \frac{2(v^2-1)}{v^2+1} \sum_{j=1}^{\infty} \left(\frac{2v}{v^2+1} \right)^{2j-2} s^{2j} \\
&\geqslant (v^2+1) + \frac{2(v^2-1)}{v^2+1}\, s^2 + \frac{2(v^2-1)}{v^2+1} \left(\frac{2v}{v^2+1} \right)^{2} \frac{s^4}{1-s^2}
\end{aligned}
$$

since $v^2 - 1 \leqslant 0$. This gives

$$
\begin{aligned}
J'(v) \geqslant \frac{2v}{v^2+1} \Bigg(& \int_{-1}^{1} (1-s^2)^{\frac{d-3}{2}}\, ds + \frac{2(v^2-1)}{(v^2+1)^2} \int_{-1}^{1} s^2 (1-s^2)^{\frac{d-3}{2}}\, ds \\
& + \frac{2(v^2-1)}{(v^2+1)^2} \left(\frac{2v}{v^2+1} \right)^2 \int_{-1}^{1} s^4 (1-s^2)^{\frac{d-5}{2}}\, ds \Bigg) \\
= \frac{2v}{v^2+1} \Bigg(& B(\tfrac{1}{2}, \tfrac{d-1}{2}) + \frac{2(v^2-1)}{(v^2+1)^2}\, B(\tfrac{3}{2}, \tfrac{d-1}{2}) + \frac{v^2-1}{v^2+1}\, \frac{8v^2}{(v^2+1)^3}\, B(\tfrac{5}{2}, \tfrac{d-3}{2}) \Bigg).
\end{aligned}
$$

Using (19) we find for all dimensions $d \geqslant 4$

$$
B(\tfrac{1}{2}, \tfrac{d-1}{2}) = d B(\tfrac{3}{2}, \tfrac{d-1}{2}) \qquad \text{and} \qquad B(\tfrac{5}{2}, \tfrac{d-3}{2}) = \frac{3}{d-3}\, B(\tfrac{3}{2}, \tfrac{d-1}{2}),
$$

and so

$$
\begin{aligned}
J'(v) &\geqslant \frac{2v}{v^2+1}\, B(\tfrac{3}{2}, \tfrac{d-1}{2}) \left(d + \frac{2(v^2-1)}{(v^2+1)^2} + \frac{3}{d-3}\, \frac{8v^2(v^2-1)}{(v^2+1)^4} \right) \\
&\geqslant \frac{2v}{v^2+1}\, B(\tfrac{3}{2}, \tfrac{d-1}{2}) \left(4 + \frac{2(v^2-1)}{(v^2+1)^2} + \frac{24v^2(v^2-1)}{(v^2+1)^4} \right).
\end{aligned}
$$

It is now straightforward to check that

$$
4 + \frac{2(v^2-1)}{(v^2+1)^2} + \frac{24v^2(v^2-1)}{(v^2+1)^4} \geqslant 0
$$

for all $v \in \mathbb{R}$. $\square$

Lemma 8. *Let $d \geqslant 3$, $0 < \alpha < 2$. There exists some $\gamma = \gamma(\alpha, d) > 0$ such that*

$$
\text{v.p.}\int_{\mathbb{R}^d} \left(\frac{1}{(|p+\lambda y|^2+1)^\gamma} - \frac{1}{(|p|^2+1)^\gamma} \right) \frac{dy}{|y|^{d+\alpha}} < 0 \tag{21}
$$

holds for all $p \in \mathbb{R}^d$, $\lambda \in \mathbb{R}$.

Proof. With the reasoning following Lemma 5 it is clear that the integral (21) exists. Without loss of generality we may assume that $\lambda = 1$. Denote the left-hand side of (21) by $I(\gamma)$. Changing to polar coordinates we get

$$I(\gamma) = \iint\limits_{S^{d-2}\times(0+,\infty)} Z(r)\, r^{-1-\alpha}\, dr d\theta = |S^{d-2}| \int\limits_{0+}^{\infty} Z(r)\, r^{-1-\alpha}\, dr,$$

(in the sense of an improper integral at the lower limit $0+$) where

$$Z(r) = \int\limits_{-1}^{1} \left(\frac{1}{(r^2+|p|^2+2r|p|s+1)^\gamma} - \frac{1}{(|p|^2+1)^\gamma} \right) (1-s^2)^{\frac{d-3}{2}}\, ds.$$

Write $Z(r) = |p|^{-2\gamma} \tilde{Z}(r)$ and observe that with $v = r/|p|$

$$\tilde{Z}(r) = \int\limits_{-1}^{1} \left(\frac{1}{(v^2+1+2vs+|p|^{-2})^\gamma} - \frac{1}{(1+|p|^{-2})^\gamma} \right) (1-s^2)^{\frac{d-3}{2}}\, ds.$$

An application of Lemma 7 with $a = 1 + |p|^{-2}$ implies

$$\begin{aligned} \left.\frac{\partial \tilde{Z}(r)}{\partial \gamma}\right|_{\gamma=0} &= -\int\limits_{-1}^{1} \big(\ln(v^2+2vs+a) - \ln(a)\big)\, (1-s^2)^{\frac{d-3}{2}}\, ds \\ &= -\Big(J(v) - \ln(a)\, I_{\frac{d-3}{2}}\Big) < 0, \end{aligned}$$

and therefore

$$I'(0) = -|p|^{-\alpha-2\gamma} \int\limits_{0+}^{\infty} \Big(J(v) - \ln(a) I_{\frac{d-3}{2}}\Big)\, v^{-1-\alpha}\, dv < 0.$$

Since $I(0) = 0$, the claim follows. ☐

Assertion (iv) of Lemma 4 follows finally from

Corollary 9. *Let $d \geqslant 3$ and $0 < \alpha < 2$. Then there exists some $\gamma = \gamma(\alpha, d) > 0$ such that for all $B \in \mathbb{R}^{d\times d}$, $b > 0$, $p \in \mathbb{R}^d$*

$$\text{v.p.}\int\limits_{\mathbb{R}^d} \left(\frac{1}{(|p+By|^2+b)^\gamma} - \frac{1}{(|p|^2+b)^\gamma} \right) \frac{dy}{|y|^{d+\alpha}} \leqslant 0. \tag{22}$$

Proof. An argument similar to the one used in the proof of Lemma 8 shows that the integral (22) is well-defined for every $\gamma > 0$. Since

$$\begin{aligned} &\text{v.p.}\int\limits_{\mathbb{R}^d} \left(\frac{1}{(|p+By|^2+b)^\gamma} - \frac{1}{(|p|^2+b)^\gamma} \right) \frac{dy}{|y|^{d+\alpha}} \\ &\quad = \frac{1}{b^\gamma}\, \text{v.p.}\int\limits_{\mathbb{R}^d} \left(\frac{1}{(|b^{-1/2}p + b^{-1/2}By|^2+1)^\gamma} - \frac{1}{(|b^{-1/2}p|^2+1)^\gamma} \right) \frac{dy}{|y|^{d+\alpha}}, \end{aligned}$$

we may assume that $b = 1$. Depending on the rank of the matrix B we distinguish between three cases.

Case 1: $\operatorname{rank} B = 0$. Nothing is to prove in this case.

Case 2: $\operatorname{rank} B = d$. We have

$$\begin{aligned}
\mathcal{J}(\lambda) &= \text{v.p.}\int\limits_{\mathbb{R}^d} \left(\frac{1}{(|p+By|^2+1)^\gamma} - \frac{1}{(|p+\lambda y|^2+1)^\gamma} \right) \frac{dy}{|y|^{d+\alpha}} \\
&= \lambda^\alpha \, \text{v.p.}\int\limits_{\mathbb{R}^d} \left(\frac{1}{(|p+\lambda^{-1}By|^2+1)^\gamma} - \frac{1}{(|p+y|^2+1)^\gamma} \right) \frac{dy}{|y|^{d+\alpha}}
\end{aligned}$$

and, therefore,

$$\lim_{\lambda\to 0} \lambda^{-\alpha}\mathcal{J}(\lambda) < 0 \qquad \text{and, by Lemma 8,} \qquad \lim_{\lambda\to\infty} \lambda^{-\alpha}\mathcal{J}(\lambda) > 0.$$

Since $\mathcal{J}(\lambda)$ is a continuous function, there exists some $\lambda^* = \lambda^*(p, B)$ such that $\mathcal{J}(\lambda^*) = 0$. Thus,

$$\begin{aligned}
&\text{v.p.}\int\limits_{\mathbb{R}^d} \left(\frac{1}{(|p+By|^2+1)^\gamma} - \frac{1}{(|p+\lambda y|^2+1)^\gamma} \right) \frac{dy}{|y|^{d+\alpha}} \\
&\qquad = \mathcal{J}(\lambda^*) + \text{v.p.}\int\limits_{\mathbb{R}^d} \left(\frac{1}{(|p+\lambda^* y|^2+1)^\gamma} - \frac{1}{(|p|^2+1)^\gamma} \right) \frac{dy}{|y|^{d+\alpha}} \leqslant 0,
\end{aligned}$$

where we used Lemma 8 again.

Case 3: $\operatorname{rank} B = k$, $1 < k < d$. In this case we can find an orthogonal matrix $S \in \mathbb{R}^{d\times d}$ such that

$$B = S \begin{pmatrix} B' & 0 \\ 0 & 0 \end{pmatrix} S^T$$

where $\tilde{B} \in \mathbb{R}^{k\times k}$ has full rank. Since the measure $|y|^{-d-\alpha}\,dy$ is invariant under orthogonal transformations we can assume that B is already of the form $\begin{pmatrix} B' & 0 \\ 0 & 0 \end{pmatrix}$; otherwise we would make a change of variables in (22) with $p' = Sp$ in place of p. Write $y = (y_1, y_2) \in \mathbb{R}^k \times \mathbb{R}^{d-k}$, $p = (p_1, p_2) \in \mathbb{R}^k \times \mathbb{R}^{d-k}$ and set $b = 1 + |p_2|^2$. Then

$$\begin{aligned}
&\text{v.p.}\int\limits_{\mathbb{R}^d} \left(\frac{1}{(|p+By|^2+1)^\gamma} - \frac{1}{(|p+\lambda y|^2+1)^\gamma} \right) \frac{dy}{|y|^{d+\alpha}} \\
&= \text{v.p.}\iint\limits_{\mathbb{R}^d} \left(\frac{1}{(|p_1+B'y_1|^2+b)^\gamma} - \frac{1}{(|p_1|^2+b)^\gamma} \right) \frac{dy_1\,dy_2}{(|y_1|^2+|y_2|^2)^{\frac{d+\alpha}{2}}} \\
&= \text{v.p.}\int\limits_{\mathbb{R}^k} \left(\frac{1}{(|p_1+B'y_1|^2+b)^\gamma} - \frac{1}{(|p_1|^2+b)^\gamma} \right) \int\limits_{\mathbb{R}^{d-k}} \frac{dy_2}{(|y_1|^2+|y_2|^2)^{\frac{d+\alpha}{2}}}\, dy_1 \\
&= \int\limits_{\mathbb{R}^{d-k}} \frac{d\eta_2}{(1+|\eta_2|^2)^{\frac{d+\alpha}{2}}}\ \text{v.p.}\int\limits_{\mathbb{R}^k} \left(\frac{1}{(|p_1+B'y_1|^2+b)^\gamma} - \frac{1}{(|p_1|^2+b)^\gamma} \right) \frac{dy_1}{|y_1|^{k+\alpha}}
\end{aligned}$$

where we used the change of variables $|y_1|\eta_2 = y_2$ in the last step. Since B' has full rank, the claim follows from case 2. $\square$

Acknowledgement. Financial support for A. Tyukov through the NTU Research Enhancement Grant *RF 175* and for R. Schilling by the Nuffield Foundation under grant *NAL/00056/G* is gratefully acknowledged. Part of the work was done under EPSRC grant *GR/R200892/01* supporting R. Schilling and A. Tyukov. We thank A. Truman and N. Jacob from the University of Wales (Swansea) for their support and valuable discussions.

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